Stability of the embeddability under perturbations of the CR structure for compact CR manifolds

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Abstract

We study the stability of the embeddability of compact 2-concave CR manifolds in complex manifolds under small horizontal perturbations of the CR structure.

The study of local and global embeddability of CR manifolds in complex manifolds has occupied a large number of mathematicians in the last forty years. Most of the results concern the case of strictly pseudoconvex CR manifolds of hypersurface type, very few is known in the other cases.

The stability of the embeddability property of a CR manifold M was first studied by N. Tanaka, [14], for strictly pseudoconvex CR manifolds of hypersurface type and real dimension greater or equal to 5 embedded in some $\mathbb{C}^N$. Few years later R. S. Hamilton, [5] et [6], was interested in the stability of the embeddability property for hypersurfaces provided the perturbation of the original CR structure is the restriction of a perturbation of a complex structure on some complex manifold X of which M is the boundary. It was proved in [8] and [7] that this last condition is satisfied for 2-concave hypersurfaces with a perturbation preserving the contact structure.

Here we consider CR manifolds of higher codimension type with mixed Levi signature and we are interested in the stability of the embeddability of such CR manifolds in complex manifolds under small perturbations of the CR structure preserving the complex tangent bundle.

Our main result is the following theorem:

Theorem 1. *Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class $\mathcal{C}^\infty$, which is smoothly embeddable as a CR manifold in a complex manifold X. Assume that $(\mathbb{M}, H_{0,1}\mathbb{M})$ is 2-concave. Let $\mathcal{E}_0$ be a smooth embedding from $\mathbb{M}$ into X and denote by M_0 the image of $\mathbb{M}$ by the embedding $\mathcal{E}_0$. Assume that either the $\overline{\partial}_b$-group of cohomology $H^{0,1}(M_0, T_{1,0}X_{|M_0}) = 0$ or $X = \mathbb{C}P^n$, and let $\widehat{H}_{0,1}\mathbb{M}$ be an horizontal perturbation of $H_{0,1}\mathbb{M}$ defined by a $(0,1)$-form $\Phi \in \mathcal{C}^{l+3}_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$, $l \geq 1$.*

Then there exists a positive real number δ such that if $\|\Phi\|_{l+3} < \delta$, then the CR manifold $(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ is embeddable in X as a CR submanifold of class $\mathcal{C}^l$.

Looking to the proof (see section 4), one can easily see that $\mathbb{C}P^n$ can be replaced by $\mathbb{C}^N \times \mathbb{C}P^{n_1} \times \cdots \times \mathbb{C}P^{n_k}$ in Theorem 1

In the case of a general complex manifold X, Theorem 1 was proved by P. L. Polyakov in [12] under the stronger hypothesis that $\mathbb{M}$ is 3-concave, and $\mathcal{E}_0$ is a generic embedding, but with a larger loss of regularity. We have to notice that, compare to our situation, Polyakov has no restriction on the kind of the perturbation of the CR structure. For example let us consider the CR manifold $M = \{(z, \zeta) \in \mathbb{C}P^n \times \mathbb{C}P^n \mid \sum_{j=1}^{n+1} z_j \overline{\zeta}_j = 0\}$ which is 2-concave if $n \geq 3$, then by Theorem 1 any sufficiently small horizontal CR perturbation of $\mathbb{M}$ is still embeddable in $\mathbb{C}P^n \times \mathbb{C}P^n$, but, if $n \geq 4$, $\mathbb{M}$ satisfies the vanishing cohomological condition (see [12]), and is 3-concave, so by Polyakov's result any sufficiently small CR perturbation (without any restriction) of $\mathbb{M}$ is still embeddable in $\mathbb{C}P^n \times \mathbb{C}P^n$.

Note that our hypothesis of horizontality of the perturbation allows us to work with anisotropic Hölder spaces and to avoid a Nash-Moser process in the proof of the theorem.

Finally note that in the case of a Levi non degenerate hypersurface, by a theorem of Gray [4] saying that all contact structures on a compact manifold near a fixed contact structure are equivalent, any perturbation of the CR structure can be reduced to an horizontal one (in that case the horizontal perturbations are the perturbations which preserve the natural contact structure). So we can immediately derive from Theorem 1 the following sharp corollary

Corollary 2. *Let $\mathbb{M}$ be a 2-concave non degenerate real quadric of $\mathbb{C}P^n$ defined by*

$$M = \{z \in \mathbb{C}P^n \mid \sum_{j=0}^{p} |z_j|^2 - \sum_{j=1}^{q} |z_{p+j}|^2 = 0\},$$

with $p \geq 2$, $q \geq 3$ and $p + q = n$. If we equip $\mathbb{M}$ with a sufficiently small perturbation of class $\mathcal{C}^{l+3}$ of the CR structure induced by the complex structure of $\mathbb{C}P^n$, then the new CR manifold is still embeddable in $\mathbb{C}P^n$ as a CR submanifold of class $\mathcal{C}^l$.

The case of the 1-concave non degenerate quadrics was studied by Biquard in [2]. He proved that the space of obstruction to the stability of the embeddability is infinite dimensional.

The paper is organized as follows:

Sections 1 and 2 consist in the description of the general setting and in the definitions of the the main objects used in this paper.

Section 3 is devoted to the proof of Theorem 1 in the general case of a complex manifold. We first remark that the problem can be reduced to the solvability of some tangential Cauchy-Riemann equation for the perturbed structure. Using global homotopy formulas with good estimates we are lead to a fixed point theorem, which gives the solution.

In section 4, we consider the case when $X = \mathbb{C}P^n$.

1   CR Structures

Let $\mathbb{M}$ be a $\mathcal{C}^l$-smooth, $l \geq 1$, paracompact differential manifold, we denote by $T\mathbb{M}$ the tangent bundle of $\mathbb{M}$ and by $T_{\mathbb{C}}\mathbb{M} = \mathbb{C} \otimes T\mathbb{M}$ the complexified tangent bundle.

Definition 1.1. An *almost CR structure* on $\mathbb{M}$ is a subbundle $H_{0,1}\mathbb{M}$ of $T_{\mathbb{C}}\mathbb{M}$ such that $H_{0,1}\mathbb{M} \cap \overline{H_{0,1}\mathbb{M}} = \{0\}$.

If the almost CR structure is integrable, i.e. for all $Z, W \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$ then $[Z, W] \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$, then it is called a *CR structure*.

If $H_{0,1}\mathbb{M}$ is a CR structure, the pair $(\mathbb{M}, H_{0,1}\mathbb{M})$ is called an *abstract CR manifold*.

The *CR dimension* of $\mathbb{M}$ is defined by CR-dim $\mathbb{M} = \operatorname{rk}_{\mathbb{C}} H_{0,1}\mathbb{M}$.

We set $H_{1,0}\mathbb{M} = \overline{H_{0,1}\mathbb{M}}$ and $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$.

Definition 1.2. Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ and $(\mathbb{M}', H_{0,1}\mathbb{M}')$ be two CR manifolds and $F : \mathbb{M} \to \mathbb{M}'$ a $\mathcal{C}^1$-map. The map F is called a *CR map* if and only if for each $x \in \mathbb{M}$, $dF((H_{0,1}\mathbb{M})_x) \subset (H_{0,1}\mathbb{M}')_{F(x)}$.

In particular, if $(\mathbb{M}, H_{0,1}\mathbb{M})$ is a CR manifold and f a complex valued function, then f is a CR function if and only if for any $\overline{L} \in H_{0,1}\mathbb{M}$ we have $\overline{L}f = 0$.

We denote by $H^{0,1}\mathbb{M}$ the dual bundle $(H_{0,1}\mathbb{M})^*$ of $H_{0,1}\mathbb{M}$. Let $\Lambda^{0,q}\mathbb{M} = \bigwedge^q(H^{0,1}\mathbb{M})$, then $\mathcal{C}^s_{0,q}(\mathbb{M}) = \Gamma^s(\mathbb{M}, \Lambda^{0,q}\mathbb{M})$ is called the space of $(0,q)$-forms of class $\mathcal{C}^s$, $0 \leq s \leq l$ on $\mathbb{M}$.

If the almost CR structure is a CR structure, i.e. if it is integrable, and if $s \geq 1$, then we can define an operator

$$\overline{\partial}_b : \mathcal{C}^s_{0,q}(\mathbb{M}) \to \mathcal{C}^{s-1}_{0,q+1}(\mathbb{M}) \tag{1.1}$$

called the *tangential Cauchy-Riemann operator* by setting $\overline{\partial}_b f = df|_{H_{0,1}\mathbb{M} \times \cdots \times H_{0,1}\mathbb{M}}$. It satisfies $\overline{\partial}_b \circ \overline{\partial}_b = 0$ and a complex valued function f is a CR function if and only if $\overline{\partial}_b f = 0$.

For $X \in H\mathbb{M}$, let $X^{0,1}$ denote the projection of X on $H_{0,1}\mathbb{M}$ and $X^{1,0}$ its projection on $H_{1,0}\mathbb{M}$.

For $(0,q)$-forms with values in the vector bundle $H_{1,0}\mathbb{M}$, we define the tangential Cauchy-Riemann operator by setting

$$\overline{\partial}_b u(\overline{L}) = [\overline{L}, u]^{1,0},$$

if $u \in \Gamma(\mathbb{M}, H_{1,0}\mathbb{M})$ and $\overline{L} \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$ and

$$\overline{\partial}_b \theta(\overline{L}_1, \ldots, \overline{L}_{q+1}) = \sum_{j=1}^{q+1} (-1)^{j+1} \overline{\partial}_b(\theta(\overline{L}_1, \ldots, \widehat{\overline{L}_j}, \ldots, \overline{L}_{q+1}))(\overline{L}_j)$$

$$+ \sum_{j<k} (-1)^{j+k} \theta([\overline{L}_j, \overline{L}_k], \overline{L}_1, \ldots, \widehat{\overline{L}_j}, \ldots, \widehat{\overline{L}_k}, \ldots, \overline{L}_{q+1}),$$

if $\theta \in \mathcal{C}^\infty_{0,q}(\mathbb{M}, H_{1,0}\mathbb{M})$ and $\overline{L}_1, \ldots, \overline{L}_{q+1} \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$. In particular for $q = 1$, we get

$$\overline{\partial}_b \theta(\overline{L}_1, \overline{L}_2) = [\overline{L}_1, \theta(\overline{L}_2)]^{1,0} - [\overline{L}_2, \theta(\overline{L}_1)]^{1,0} - \theta([\overline{L}_1, \overline{L}_2]).$$

The annihilator $H^0\mathbb{M}$ of $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ in $T_{\mathbb{C}}^*\mathbb{M}$ is called the *characteristic bundle* of $\mathbb{M}$. Given $p \in \mathbb{M}$, $\omega \in H^0_p\mathbb{M}$ and $X, Y \in H_p\mathbb{M}$, we choose $\widetilde{\omega} \in \Gamma(\mathbb{M}, H^0\mathbb{M})$ and

$\widetilde{X}, \widetilde{Y} \in \Gamma(\mathbb{M}, H\mathbb{M})$ with $\widetilde{\omega}_p = \omega$, $\widetilde{X}_p = X$ and $\widetilde{Y}_p = Y$. Then $d\widetilde{\omega}(X, Y) = -\omega([\widetilde{X}, \widetilde{Y}])$. Therefore we can associate to each $\omega \in H_p^0\mathbb{M}$ an hermitian form

$$L_\omega(X) = -i\omega([\widetilde{X}, \overline{\widetilde{X}}]) \tag{1.2}$$

on $H_p\mathbb{M}$. This is called the *Levi form* of $\mathbb{M}$ at $\omega \in H_p^0\mathbb{M}$.

In the study of the $\overline{\partial}_b$-complex two important geometric conditions were introduced for CR manifolds of real dimension $2n - k$ and CR-dimension $n - k$. The first one by Kohn in the hypersurface case, $k = 1$, the condition Y(q), the second one by Henkin in codimension k, $k \geq 1$, the q-concavity.

A CR manifold $\mathbb{M}$ satisfies Kohn's condition $Y(q)$ at a point $p \in \mathbb{M}$ for some $0 \leq q \leq n - 1$, if the Levi form of $\mathbb{M}$ at p has at least $\max(n - q, q + 1)$ eigenvalues of the same sign or at least $\min(n - q, q + 1)$ eigenvalues of opposite signs.

A CR manifold $\mathbb{M}$ is said to be *q-concave* at $p \in \mathbb{M}$ for some $0 \leq q \leq n - k$, if the Levi form L_ω at $\omega \in H_p^0\mathbb{M}$ has at least q negative eigenvalues on $H_p\mathbb{M}$ for every nonzero $\omega \in H_p^0\mathbb{M}$.

In [13] the condition Y(q) is extended to arbitrary codimension.

Definition 1.3. An abstract CR manifold is said to satisfy *condition Y(q)* for some $1 \leq q \leq n - k$ at $p \in \mathbb{M}$ if the Levi form L_ω at $\omega \in H_p^0\mathbb{M}$ has at least $n - k - q + 1$ positive eigenvalues or at least $q + 1$ negative eigenvalues on $H_p\mathbb{M}$ for every nonzero $\omega \in H_p^0\mathbb{M}$.

Note that in the hypersurface type case, i.e. $k = 1$, this condition is equivalent to the classical condition Y(q) of Kohn for hypersurfaces and in particular if the CR structure is strictly peudoconvex, i.e. the Levi form is positive definite or negative definite, condition Y(q) holds for all $1 \leq q < n - 1$. Moreover, if $\mathbb{M}$ is q-concave at $p \in \mathbb{M}$, then $q \leq (n - k)/2$ and condition Y(r) is satisfied at $p \in \mathbb{M}$ for any $0 \leq r \leq q - 1$ and $n - k - q + 1 \leq r \leq n - k$.

Definition 1.4. Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract CR manifold, X be a complex manifold and $F : \mathbb{M} \to X$ be an embedding of class $\mathcal{C}^l$, then F is called a *CR embedding* if $dF(H_{0,1}\mathbb{M})$ is a subbundle of the bundle $T_{0,1}X$ of the holomorphic vertor fields of X and $dF(H_{0,1}\mathbb{M}) = T_{0,1}X \cap T_{\mathbb{C}}F(\mathbb{M})$.

Let F be a CR embedding of an abstract CR manifold into a complex manifold X and set $M = F(\mathbb{M})$, then M is a CR manifold with the CR structure $H_{0,1}M = T_{0,1}X \cap T_{\mathbb{C}}M$.

Let U be a coordinate domain in X, then $F_{|_{F^{-1}(U)}} = (f_1, \ldots, f_N)$, with $N = \dim_{\mathbb{C}}X$, and F is a CR embedding if and only if, for all $1 \leq j \leq N$, $\overline{\partial}_b f_j = 0$.

A CR embedding is called *generic* if $\dim_{\mathbb{C}}X - \mathrm{rk}_{\mathbb{C}}H_{0,1}M = \mathrm{codim}_{\mathbb{R}}M$.

2   Perturbation of CR structures

In this section we shall define the notion of perturbation of a given CR structure on a manifold and introduce some new complex associated to the perturbed structure.

Definition 2.1. An almost CR structure $\widehat{H}_{0,1}\mathbb{M}$ on $\mathbb{M}$ is said to be *a perturbation of finite distance* to a given CR structure $H_{0,1}\mathbb{M}$ if $\widehat{H}_{0,1}\mathbb{M}$ can be represented as a graph in $T_{\mathbb{C}}\mathbb{M}$

over $H_{0,1}\mathbb{M}$. It is called an *horizontal perturbation* of the CR structure $H_{0,1}\mathbb{M}$ if it can be represented as a graph in the complex tangent bundle $H\mathbb{M} = H_{1,0}\mathbb{M} \oplus H_{0,1}\mathbb{M}$ over $H_{0,1}\mathbb{M}$, which means that there exists $\Phi \in \mathcal{C}_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$ such that

$$\widehat{H}_{0,1}\mathbb{M} = \{\overline{W} \in T_{\mathbb{C}}\mathbb{M} \mid \overline{W} = \overline{Z} - \Phi(\overline{Z}), \overline{Z} \in H_{0,1}\mathbb{M}\}. \tag{2.1}$$

The horizontal perturbation $\widehat{H}_{0,1}\mathbb{M}$ of the CR structure $H_{0,1}\mathbb{M}$ will be integrable if and only if given $\overline{L}_1, \overline{L}_2 \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$ and $\overline{L}_i^{\Phi} = \overline{L}_i - \Phi(\overline{L}_i)$, $i = 1, 2$, there exists $\overline{W}$ such that $[\overline{L}_1^{\Phi}, \overline{L}_2^{\Phi}] = \overline{W} - \Phi(\overline{W}) = \overline{W}^{\Phi}$, which is equivalent to

$$[\overline{L}_1^{\Phi}, \overline{L}_2^{\Phi}]^{1,0} = -\Phi([\overline{L}_1^{\Phi}, \overline{L}_2^{\Phi}]^{0,1}), \tag{2.2}$$

if $X^{0,1}$ denotes the projection on $H_{0,1}\mathbb{M}$ and $X^{1,0}$ the projection on $H_{1,0}\mathbb{M}$ of $X \in H\mathbb{M}$.

Since

$$\begin{aligned}
[\overline{L}_1^{\Phi}, \overline{L}_2^{\Phi}] &= [\overline{L}_1, \overline{L}_2] - [\Phi(\overline{L}_1), \overline{L}_2] - [\overline{L}_1, \Phi(\overline{L}_2)] + [\Phi(\overline{L}_1), \Phi(\overline{L}_2)] \\
&= [\overline{L}_1, \overline{L}_2] - \Phi([\overline{L}_1, \overline{L}_2]) - ([\overline{L}_1, \Phi(\overline{L}_2)] - [\overline{L}_2, \Phi(\overline{L}_1)] - \Phi([\overline{L}_1, \overline{L}_2])) \\
&\quad + [\Phi(\overline{L}_1), \Phi(\overline{L}_2)]
\end{aligned}$$

we get

$$[\overline{L}_1^{\Phi}, \overline{L}_2^{\Phi}]^{1,0} = -\Phi([\overline{L}_1, \overline{L}_2]) - \overline{\partial}_b \Phi(\overline{L}_1, \overline{L}_2) + [\Phi(\overline{L}_1), \Phi(\overline{L}_2)]$$

and the equation (2.2) is equivalent to

$$\overline{\partial}_b \Phi(\overline{L}_1, \overline{L}_2) = [\Phi(\overline{L}_1), \Phi(\overline{L}_2)] - \Phi([\overline{L}_1, \Phi(\overline{L}_2)]^{0,1}) + \Phi([\overline{L}_2, \Phi(\overline{L}_1)]^{0,1}),$$

which we simply write in the form

$$\overline{\partial}_b^{H_{1,0}\mathbb{M}} \Phi = \frac{1}{2}[\Phi, \Phi]. \tag{2.3}$$

Note that if $\widehat{H}_{0,1}\mathbb{M}$ is an integrable horizontal perturbation of $H_{0,1}\mathbb{M}$, the space $H\widehat{\mathbb{M}} = H_{1,0}\widehat{\mathbb{M}} \oplus H_{0,1}\widehat{\mathbb{M}}$ coincides with the space $H\mathbb{M}$ and consequently the two abstract CR manifolds $\mathbb{M}$ and $\widehat{\mathbb{M}}$ have the same characteristic bundle and hence the same Levi form. This implies in particular that if $\mathbb{M}$ satisfies condition Y(q) at each point, then $\widehat{\mathbb{M}}$ satisfies also condition Y(q) at each point and that if $\mathbb{M}$ is q-concave then $\widehat{\mathbb{M}}$ is also q-concave.

Assume $\mathbb{M}$ is an abstract CR manifold and $\widehat{H}_{0,1}\mathbb{M}$ is an integrable horizontal perturbation of the original CR structure $H_{0,1}\mathbb{M}$ on $\mathbb{M}$. Let f be a complex valued function on $\mathbb{M}$, then f will be CR for the new structure $\widehat{H}_{0,1}\mathbb{M}$ if and only if

$$\forall \overline{L}^{\Phi} \in \Gamma(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M}), \ \overline{L}^{\Phi} f = 0 \ \Leftrightarrow \ \overline{\partial}_b^{\Phi} f = df|_{\widehat{H}_{0,1}\mathbb{M}} = 0.$$

But going back to the definition of $\widehat{H}_{0,1}\mathbb{M}$, this means

$$\forall \overline{L} \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M}), \ (\overline{L} - \Phi(\overline{L}))f = 0 \ \Leftrightarrow \ d_{\Phi}'' f = (\overline{\partial}_b - \Phi \lrcorner \partial_b)f = 0,$$

where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator associated to the original CR structure $H_{0,1}\mathbb{M}$ and $\Phi \lrcorner \partial_b f$ is the $(0,1)$-form for the initial structure defined by $\Phi \lrcorner \partial_b f(\overline{L}) = \Phi(\overline{L})f$ for any section $\overline{L} \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$.

Let us consider the two operators

$$\overline{\partial}_b^{\Phi} : \mathcal{C}^1(\mathbb{M}) \to \Gamma^0(\mathbb{M}, \widehat{H}^{0,1}\mathbb{M}) \quad \text{and} \quad d_{\Phi}'' : \mathcal{C}^1(\mathbb{M}) \to \Gamma^0(\mathbb{M}, H^{0,1}\mathbb{M}),$$

then a fonction f is CR for the new structure $\widehat{H}_{0,1}\mathbb{M}$ if and only if $\overline{\partial}_b^{\Phi} f = 0$ or equivalently $d_{\Phi}'' f = 0$.

We are lead to consider two pseudo-complex :

$$(\mathcal{C}_{0,*}^{\infty}(\widehat{\mathbb{M}}), \overline{\partial}_b^{\Phi}) \quad \text{and} \quad (\mathcal{C}_{0,*}^{\infty}(\mathbb{M}), d_{\Phi}''),$$

where $\mathcal{C}_{0,q}^{\infty}(\widehat{\mathbb{M}}) = \Gamma(\mathbb{M}, \bigwedge^q(\widehat{H}^{0,1}\mathbb{M}))$ and $\mathcal{C}_{0,q}^{\infty}(\mathbb{M}) = \Gamma(\mathbb{M}, \bigwedge^q(H^{0,1}\mathbb{M}))$. In degree $q \geq 1$, since the two structures $H^{0,1}\mathbb{M}$ and $\widehat{H}^{0,1}\mathbb{M}$ are integrable, the operators can be defined in the following way : if $\alpha \in \mathcal{C}_{0,q}^{\infty}(\widehat{\mathbb{M}})$ and $\overline{L}_1^{\Phi}, \ldots, \overline{L}_{q+1}^{\Phi} \in \Gamma(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$, we set

$$\overline{\partial}_b^{\Phi} \alpha(\overline{L}_1^{\Phi}, \ldots, \overline{L}_{q+1}^{\Phi}) = \sum_{j=1}^{q+1} (-1)^{j+1} \overline{\partial}_b^{\Phi}(\alpha(\overline{L}_1^{\Phi}, \ldots, \widehat{\overline{L}_j^{\Phi}}, \ldots, \overline{L}_{q+1}^{\Phi}))(\overline{L}_j^{\Phi})$$

$$+ \sum_{j<k} (-1)^{j+k} \alpha([\overline{L}_j^{\Phi}, \overline{L}_k^{\Phi}], \overline{L}_1^{\Phi}, \ldots, \widehat{\overline{L}_j^{\Phi}}, \ldots, \widehat{\overline{L}_k^{\Phi}}, \ldots, \overline{L}_{q+1}^{\Phi})$$

and if $\beta \in \mathcal{C}_{0,q}^{\infty}(\mathbb{M})$ and $\overline{L}_1, \ldots, \overline{L}_{q+1} \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$, we set

$$d_{\Phi}'' \beta(\overline{L}_1, \ldots, \overline{L}_{q+1}) = \sum_{j=1}^{q+1} (-1)^{j+1} d_{\Phi}''(\beta(\overline{L}_1, \ldots, \widehat{\overline{L}_j}, \ldots, \overline{L}_{q+1}))(\overline{L}_j)$$

$$+ \sum_{j<k} (-1)^{j+k} \beta([\overline{L}_j, \overline{L}_k], \overline{L}_1, \ldots, \widehat{\overline{L}_j}, \ldots, \widehat{\overline{L}_k}, \ldots, \overline{L}_{q+1}).$$

Note that $\overline{\partial}_b^{\Phi} = d|_{\widehat{H}_{0,1}\mathbb{M} \times \cdots \times \widehat{H}_{0,1}\mathbb{M}}$ and the pseudo-complex $(\mathcal{C}_{0,*}^{\infty}(\widehat{\mathbb{M}}), \overline{\partial}_b^{\Phi})$ is a differential complex, which is nothing else than the tangential Cauchy-Riemann complex on $\mathbb{M}$ associated to the new CR structure $\widehat{H}^{0,1}\mathbb{M}$.

Lemma 2.2. *The pseudo-complex $(\mathcal{C}_{0,*}^{\infty}(\mathbb{M}), d_{\Phi}'')$ is a differential complex, i.e. $d_{\Phi}'' \circ d_{\Phi}'' = 0$ if and only if Φ satisfies $[\overline{L}_1^{\Phi}, \overline{L}_2^{\Phi}] = [\overline{L}_1, \overline{L}_2]^{\Phi}$, for any $\overline{L}_1, \overline{L}_2 \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$.*

Proof. Let $f \in \mathcal{C}^{\infty}(\mathbb{M})$ be a $\mathcal{C}^{\infty}$-smooth function on $\mathbb{M}$, going back to the definition of the operator d_{Φ}'' we get

$$d_{\Phi}'' \circ d_{\Phi}'' f(\overline{L}_1, \overline{L}_2) = \overline{L}_1^{\Phi} \overline{L}_2^{\Phi} f - \overline{L}_2^{\Phi} \overline{L}_1^{\Phi} f - [\overline{L}_1, \overline{L}_2]^{\Phi} f$$

$$= \overline{\partial}_b^{\Phi} \circ \overline{\partial}_b^{\Phi} f(\overline{L}_1^{\Phi}, \overline{L}_2^{\Phi}) + ([\overline{L}_1^{\Phi}, \overline{L}_2^{\Phi}] - [\overline{L}_1, \overline{L}_2]^{\Phi}) f$$

$$= ([\overline{L}_1^{\Phi}, \overline{L}_2^{\Phi}] - [\overline{L}_1, \overline{L}_2]^{\Phi}) f,$$

which proves the lemma. $\square$

3 Embedding of small horizontal perturbations in complex manifolds

Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class $\mathcal{C}^\infty$ and $\mathcal{E}_0 : \mathbb{M} \to M_0 \subset X$ be a $\mathcal{C}^\infty$-smooth CR embedding in a complex manifold X, then M_0 is a compact CR submanifold of X of class $\mathcal{C}^\infty$ with the CR structure $H_{0,1}M_0 = d\mathcal{E}_0(H_{0,1}\mathbb{M}) = T_\mathbb{C}M_0 \cap T_{0,1}X$ and the tangential Cauchy-Riemann operator $\overline{\partial}_b$.

Let $\widehat{H}_{0,1}\mathbb{M}$ be an horizontal perturbation of $H_{0,1}\mathbb{M}$, we are looking for an embedding $\mathcal{E} : \mathbb{M} \to M \subset X$ of class $\mathcal{C}^l$, $l \geq 1$, such that $d\mathcal{E}(\widehat{H}_{0,1}\mathbb{M}) = T_\mathbb{C}M \cap T_{0,1}X$, i.e. $\mathcal{E}$ is a CR embedding.

Set $\widehat{H}_{0,1}M_0 = d\mathcal{E}_0(\widehat{H}_{0,1}\mathbb{M})$, as $\mathcal{E}_0$ is a CR embedding then $\widehat{H}_{0,1}M_0$ is an horizontal perturbation of $H_{0,1}M_0$ and consequently it is defined by a $(0,1)$-form $\Phi \in \mathcal{C}^k_{0,1}(M_0, H_{1,0}M_0)$, $k \geq 1$. We denote by $\overline{\partial}_b^{\Phi}$ the associated tangential Cauchy-Riemann operator.

We will consider only *small* (the sense will be precised later) perturbations of the original structure, thus it is reasonable to assume that the diffeomorphism $F = \mathcal{E} \circ \mathcal{E}_0^{-1} : M_o \subset X \to M \subset X$ is close to identity.

We equip the manifold X with some Riemannian metric (for example, if $X = \mathbb{C}P^n$, take the Fubini-Study metric). The idea is to look for some F in the subset of the restrictions to M_0 of $\mathcal{C}^l$-diffeomorphisms of X parametrized by sections of the vector bundle TX by mean of the exponential map. Let us consider the following diagram where U is a neighborhood of the zero section and σ a section of TX over M_0

$$
\begin{array}{ccc}
U \subset TX & \stackrel{\exp}{\longrightarrow} & X \times X \\
(x, \sigma(x)) & \longmapsto & (x, F(x)) = \exp_x \sigma(x) \\
\uparrow & & \downarrow \\
M_0 & \longrightarrow & X \\
x & \longmapsto & F(x)
\end{array}
$$

In fact $\mathcal{E}$ will be a CR embedding if and only if F is CR as a map from $\widehat{M}_0 = (M_0, \widehat{H}_{0,1}M_0)$ into X, which means that we have to find a section σ of TX over M_0 such that the image of the new CR structure $\widehat{H}_{0,1}M_0$ by the tangent map to $\exp \circ \sigma$ is contained in $T_{0,1}(X \times X)$.

More precisely, using that for all $x \in M_0$ we can write $F(x) = \pi(x, F(x)) = \pi(\exp_x \sigma(x))$, with π the second projection from $X \times X$ onto X, the map F will be CR if and only if for all vector fields $\overline{L}^\Phi = \overline{L} - \Phi(\overline{L}) \in \Gamma(\mathbb{M}, \widehat{H}^{0,1}\mathbb{M})$, $\overline{L} \in \Gamma(\mathbb{M}, H^{0,1}\mathbb{M})$, we have

$$
dF(\overline{L}^\Phi) = d(\pi(\exp_x \sigma(x))(\overline{L}^\Phi) = 0.
$$

As the differential of the map exp at a point is given by the map $(u, \xi) \mapsto (u, u + \xi)$, this is equivalent to $(d(Id) + d(\sigma))(\overline{L}^\Phi) = 0$, i.e.

$$
\overline{\partial}_b^{\Phi}\sigma = -\overline{\partial}_b^{\Phi}Id.
$$

Since $d(Id)(\overline{L}^\Phi) = -\Phi(\overline{L})$, all that means that we have to solve the equation

$$
d''_\phi \sigma = \Phi.
$$

The remaining of the section will be devoted to the proof of the following theorem:

Theorem 3.1. *Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class $\mathcal{C}^\infty$, which is smoothly embeddable as a CR manifold in a complex manifold X. Assume that $(\mathbb{M}, H_{0,1}\mathbb{M})$ is 2-concave. Let $\mathcal{E}_0$ be a smooth embedding from $\mathbb{M}$ into X and denote by M_0 the image of $\mathbb{M}$ by the embedding $\mathcal{E}_0$. Assume the $\overline{\partial}_b$-group of cohomology $H^{0,1}(M_0, T_{1,0}X_{|M_0}) = 0$ and let $\widehat{H}_{0,1}\mathbb{M}$ be an horizontal perturbation of $H_{0,1}\mathbb{M}$ defined by a $(0,1)$-form $\Phi \in \mathcal{C}^{l+3}_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$, $l \geq 1$.*

Then there exists a positive real number δ such that if $\|\Phi\|_{l+3} < \delta$, then the CR manifold $(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ is embeddable in X as a CR submanifold of class $\mathcal{C}^l$.

3.1   Reduction to a fixed point theorem

Let E be a CR bundle over $\mathbb{M}$ which satisfies $H^{0,1}(\mathbb{M}, E) = 0$ and g be a $\overline{\partial}_b^\Phi$-closed $(0,1)$-form in $\mathcal{C}^{l+2}_{0,1}(\widehat{\mathbb{M}}, E)$ and let us consider the following equation:

$$\overline{\partial}_b^\Phi v = g. \tag{3.1}$$

By definition of the $\overline{\partial}_b^\Phi$-operator, the equation $\overline{\partial}_b^\Phi v = g$ is equivalent to the equation

$$\overline{\partial}_b v = \widetilde{g} + \Phi \lrcorner \partial_b v, \tag{3.2}$$

where $\widetilde{g}$ is the $(0,1)$-form on $\mathbb{M}$ relatively to the initial structure $H_{0,1}\mathbb{M}$ defined by $\widetilde{g}(\overline{L}) = g(\overline{L} - \Phi(\overline{L}))$ for $\overline{L} \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$.

A natural tool to solve such an equation is a global homotopy formula for the $\overline{\partial}_b$-operator with good estimates.

Assume $\mathbb{M}$ is 2-concave, then by [11], since $\mathbb{M}$ is embeddable and 1-concave, $\mathbb{M}$ is locally generically embeddable and we may apply the results in [1] and [10] on local estimates and global homotopy formulas for the tangential Cauchy-Riemann operator.

In [1] the following result is proved

Proposition 3.2. *Let M be a 2-concave CR generic submanifold of X of class $\mathcal{C}^\infty$. For each point in M, there exist a neighborhood U and linear operators*

$$T_r \; : \; \mathcal{C}^0_{n,r}(M) \to \mathcal{C}^0_{n,r-1}(U), \quad 1 \leq r \leq 2$$

with the following two properties :
(i) For all $l \in \mathbb{N}$ and $1 \leq r \leq 2$,

$$T_r(\mathcal{C}^l_{n,r}(M)) \subset \mathcal{C}^{l+1/2}_{n,r-1}(\overline{U})$$

and T_r is continuous as an operator between $\mathcal{C}^l_{n,r}(M)$ and $\mathcal{C}^{l+1/2}_{n,r-1}(\overline{U})$.
(ii) If $f \in \mathcal{C}^1_{n,r}(M)$, $0 \leq r \leq 1$, has compact support in U, then, on U,

$$f = \begin{cases} T_1 \overline{\partial}_b f & \text{if } r = 0\,, \\ \overline{\partial}_b T_1 f + T_2 \overline{\partial}_b f & \text{if } r = 1\,. \end{cases} \tag{3.3}$$

and in [10] we have derived from the previous proposition a global homotopy formula by mean of a functional analytic construction.

Theorem 3.3. *Let E be an holomorphic vector bundle over X and $\mathbb{M}$ be a compact 2-concave locally generically embeddable CR submanifold of X of class $\mathcal{C}^\infty$ such that $H^{0,1}(\mathbb{M}, E) = 0$. Then there exist continuous linear operators*

$$A_r \; : \; \mathcal{C}^0_{0,r}(\mathbb{M}, E) \to \mathcal{C}^0_{0,r-1}(\mathbb{M}, E), \quad 1 \le r \le 2$$

such that
   (i) For all $l \in \mathbb{N}$ and $1 \le r \le 2$,

$$A_r(\mathcal{C}^l_{0,r}(\mathbb{M}, E)) \subset \mathcal{C}^{l+1/2}_{0,r-1}(\mathbb{M}, E)$$

and A_r is continuous as an operator between $\mathcal{C}^l_{0,r}(\mathbb{M}, E)$ and $\mathcal{C}^{l+1/2}_{0,r-1}(\mathbb{M}, E)$.
   (ii) For all $f \in \mathcal{C}^1_{0,1}(\mathbb{M}, E)$

$$f = \overline{\partial}_b A_1 f + A_2 \overline{\partial}_b f \tag{3.4}$$

   In fact we need better estimates than the previous ones to reduce the solvability of our equation 3.2 to a fixed point theorem.
   Assume there exist Banach spaces $\mathcal{B}^l(\mathbb{M})$, $l \in \mathbb{N}$, with the following properties :
   (i) $\mathcal{C}^{l+2}(\mathbb{M}) \subset \mathcal{B}^{2l+1}(\mathbb{M}) \subset \mathcal{B}^{2l}(\mathbb{M}) \subset \mathcal{C}^l(\mathbb{M})$;
   (ii) $\cap_{l \in \mathbb{N}} \mathcal{B}^l(\mathbb{M}) = \mathcal{C}^\infty(\mathbb{M})$;
   (iii) $\mathcal{B}^l(\mathbb{M})$ is invariant under horizontal perturbations of the CR structure
   (iv) If $f \in \mathcal{B}^l(\mathbb{M})$, $l \ge 1$, $X_\mathbb{C} f \in \mathcal{B}^{l-1}(\mathbb{M})$ when $X_\mathbb{C}$ is a complex vector field tangent to $\mathbb{M}$
   and that the previous operators A_r, $r = 1, 2$ are linear continuous operators from $\mathcal{B}^l_{0,r}(\mathbb{M}, E)$ into $\mathcal{B}^{l+1}_{0,r-1}(\mathbb{M}, E)$
   Since $\mathbb{M}$ is 2-concave and E satisfies $H^{0,1}(\mathbb{M}, E) = 0$, under this additional hypothesis, if v is a solution of (3.1), then $\overline{\partial}_b(\widetilde{g} + \Phi \lrcorner \partial_b v) = 0$ and by (3.4)

$$\overline{\partial}_b(A_1(\widetilde{g} + \Phi \lrcorner \partial_b v)) = \widetilde{g} + \Phi \lrcorner \partial_b v.$$

   Assume Φ is of class $\mathcal{C}^{l+2}$, then the map

$$\Theta \; : \; \mathcal{B}^{2l+1}(\mathbb{M}, E) \to \mathcal{B}^{2l+1}(\mathbb{M}, E)$$
$$v \mapsto A_1 \widetilde{g} + A_1(\Phi \lrcorner \partial_b v) \,.$$

is continuous, and the fixed points of Θ are good candidates to be solutions of (3.1).

3.2   A fixed point theorem

In this section we assume that all the assumptions of the previous section are satisfied.
   Let δ_0 such that, if $\|\Phi\|_{l+2} < \delta_0$, then the norm of the bounded endomorphism $A_1 \circ \Phi \lrcorner \partial_b$ of $\mathcal{B}^{2l+1}(\mathbb{M}, E)$ is equal to $\epsilon_0 < 1$. We shall prove that, if $\|\Phi\|_{l+2} < \delta_0$, the map Θ admits a unique fixed point, which is a solution of the equation $\overline{\partial}^\Phi_b v = g$.
   Consider first the uniqueness of the fixed point. Assume v_1 and v_2 are two fixed points of Θ, then

$$v_1 = \Theta(v_1) = A_1 \widetilde{g} + A_1(\Phi \lrcorner \partial_b v_1)$$
$$v_2 = \Theta(v_2) = A_1 \widetilde{g} + A_1(\Phi \lrcorner \partial_b v_2).$$

This implies

$$v_1 - v_2 = A_1\big(\Phi\lrcorner\partial_b(v_1 - v_2)\big)$$

and, by the hypothesis on Φ,

$$\|v_1 - v_2\|_{\mathcal{B}^{2l+1}} < \|v_1 - v_2\|_{\mathcal{B}^{2l+1}}$$

or $v_1 = v_2$ and hence $v_1 = v_2$.

For the existence we proceed by iteration. We set $v_0 = \Theta(0) = A_1(\widetilde{g})$ and, for $n \geq 0$, $v_{n+1} = \Theta(v_n)$. Then for $n \geq 0$, we get

$$v_{n+1} - v_n = A_1(\Phi\lrcorner\partial_b(v_n - v_{n-1})).$$

Therefore, if $\|\Phi\|_{l+2} < \delta_0$, the sequence $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathcal{B}^{2l+1}(\mathbb{M}, E)$ and hence converges to a form v, moreover by continuity of the map Θ, v satisfies $\Theta(v) = v$.

It remains to prove that v is a solution of (3.1). Since $H^{0,1}(\mathbb{M}, E) = 0$, it follows from (3.4) and from the definition of the sequence $(v_n)_{n\in\mathbb{N}}$ that

$$\widetilde{g} - d''_\Phi v_{n+1} = \Phi\lrcorner\partial_b(v_{n+1} - v_n) + A_2\overline{\partial}_b(\widetilde{g} + \Phi\lrcorner\partial_b v_n).$$

For any $\overline{L}_1, \overline{L}_2 \in \Gamma(\mathbb{M}, H_{0,1}\mathbb{M})$,

$$\overline{\partial}_b(\widetilde{g} + \Phi\lrcorner\partial_b v_n)(\overline{L}_1, \overline{L}_2) = (\overline{\partial}_b\widetilde{g} - \overline{\partial}_b(\overline{\partial}_b - \Phi\lrcorner\partial_b)v_n)(\overline{L}_1, \overline{L}_2)$$
$$= \overline{\partial}_b(\widetilde{g} - d''_\Phi v_n)(\overline{L}_1, \overline{L}_2)$$

Let us consider the $(0,1)$-form u for the new structure defined by $u = g - \overline{\partial}_b^\Phi v_n$, the associated $(0,1)$-form $\widetilde{u}$ for the initial structure satisfies $\widetilde{u} = \widetilde{g} - d''_\Phi v_n$. Note that, since $(\overline{\partial}_b^\Phi)^2 = 0$ and $\overline{\partial}_b^\Phi g = 0$, we have $\overline{\partial}_b^\Phi u = 0$ and this implies

$$\overline{\partial}_b\widetilde{u}(\overline{L}_1, \overline{L}_2) = \overline{L}_1\widetilde{u}(\overline{L}_2) - \overline{L}_2\widetilde{u}(\overline{L}_1) - \widetilde{u}([\overline{L}_1, \overline{L}_2])$$
$$= \overline{L}_1^\Phi u(\overline{L}_2^\Phi) - \overline{L}_2^\Phi u(\overline{L}_1^\Phi) + \Phi(\overline{L}_1)u(\overline{L}_2^\Phi) - \Phi(\overline{L}_2)u(\overline{L}_1^\Phi) - \widetilde{u}([\overline{L}_1, \overline{L}_2])$$
$$= \overline{\partial}_b^\Phi u(\overline{L}_1^\Phi, \overline{L}_2^\Phi) + u([\overline{L}_1^\Phi, \overline{L}_2^\Phi]) - u([\overline{L}_1, \overline{L}_2] - \Phi([\overline{L}_1, \overline{L}_2]))$$
$$+ \Phi(\overline{L}_1)u(\overline{L}_2^\Phi) - \Phi(\overline{L}_2)u(\overline{L}_1^\Phi)$$
$$= u([\overline{L}_1^\Phi, \overline{L}_2^\Phi]) - [\overline{L}_1, \overline{L}_2] + \Phi([\overline{L}_1, \overline{L}_2])$$
$$+ \Phi(\overline{L}_1)u(\overline{L}_2^\Phi) - \Phi(\overline{L}_2)u(\overline{L}_1^\Phi)$$

Since both structures $H_{0,1}\mathbb{M}$ and $\widehat{H}_{0,1}\mathbb{M}$ are integrable, the vector field $[\overline{L}_1^\Phi, \overline{L}_2^\Phi]) - [\overline{L}_1, \overline{L}_2] + \Phi([\overline{L}_1, \overline{L}_2]$ is a section of $\widehat{H}_{0,1}\mathbb{M}$, hence there exists $\overline{W}_\Phi(\overline{L}_1, \overline{L}_2)$ such that

$$[\overline{L}_1^\Phi, \overline{L}_2^\Phi]) - [\overline{L}_1, \overline{L}_2] + \Phi([\overline{L}_1, \overline{L}_2] = \overline{W}_\Phi(\overline{L}_1, \overline{L}_2) - \Phi(\overline{W}_\Phi(\overline{L}_1, \overline{L}_2)).$$

Moreover

$$[\overline{L}_1^\Phi, \overline{L}_2^\Phi]) - [\overline{L}_1, \overline{L}_2] + \Phi([\overline{L}_1, \overline{L}_2] = [\Phi(\overline{L}_1), \Phi(\overline{L}_2)] - \overline{\partial}_b\Phi(\overline{L}_1, \overline{L}_2) - \Phi(\overline{L}_1)\overline{L}_2 + \Phi(\overline{L}_2)\overline{L}_1)$$

and $\overline{W}_\Phi(\overline{L}_1, \overline{L}_2)$ depends on Φ at the order 1.

Finally
$$\overline{\partial}_b \widetilde{u}(\overline{L}_1, \overline{L}_2) = \widetilde{u}(\overline{W}_\Phi(\overline{L}_1, \overline{L}_2)) + \Phi(\overline{L}_1)\widetilde{u}(\overline{L}_2) - \Phi(\overline{L}_2)\widetilde{u}(\overline{L}_1)$$

and going back to $u = g - \overline{\partial}_b^\Phi v_n$ we get

$$\widetilde{g} - d''_\Phi v_{n+1} = \Phi \lrcorner \partial_b(v_{n+1} - v_n) + A_2(\tau_\Phi(\widetilde{g} - d''_\Phi v_n)), \tag{3.5}$$

where τ_Φ is defined so that $\tau_\Phi(\widetilde{u})(\overline{L}_1, \overline{L}_2) = \widetilde{u}(\overline{W}_\Phi(\overline{L}_1, \overline{L}_2)) + \Phi(\overline{L}_1)\widetilde{u}(\overline{L}_2) - \Phi(\overline{L}_2)\widetilde{u}(\overline{L}_1)$.

Note that since $g \in \mathcal{C}_{0,1}^{l+2}(\mathbb{M}, E)$ and Φ is of class $\mathcal{C}^{l+2}$, then $v_0 \in \mathcal{B}^{2l+1}(\mathbb{M})$ and $\overline{\partial}_b v_0 - \Phi \lrcorner \partial_b v_0 = d''_\Phi v_0$ is in $\mathcal{B}_{0,1}^{2l}(\mathbb{M}, E)$, moreover, by definition of an horizontal perturbation, for any section $\overline{L}$ of $H_{0,1}\mathbb{M}$ the vector field $\Phi(\overline{L})$ is complex tangent.

Thus it follows by induction that $d''_\Phi v_n \in \mathcal{B}_{0,1}^{2l}(\mathbb{M}, E)$ for all $n \in \mathbb{N}$ and, if Φ is of class $\mathcal{C}^{l+3}$, by (3.5) we have the estimate

$$\|\widetilde{g} - d''_\Phi v_{n+1}\|_{\mathcal{B}^{2l}} \leq \|\Phi \lrcorner \partial_b\|\|(v_{n+1} - v_n)\|_{\mathcal{B}^{2l+1}} + \|A_2\|\|\widetilde{g} - d''_\Phi v_n\|_{\mathcal{B}^{2l}}\|\Phi\|_{l+3}. \tag{3.6}$$

Let δ such that if $\|\Phi\|_{l+3} < \delta$, then the maximum $\|A_1 \circ \Phi \lrcorner \partial_b\|$ and $\|\Phi\|_{l+3}\|A_2\|$ is equal to $\epsilon < 1$. Assume $\|\Phi\|_{l+3} < \delta$, then by induction we get

$$\|\widetilde{g} - d''_\Phi v_{n+1}\|_{\mathcal{B}^{2l}} \leq (n+1)\epsilon^{n+1}\|\Phi \lrcorner \partial_b\|\|v_0\|_{\mathcal{B}^{2l+1}} + \epsilon^{n+1}\|\widetilde{g} - d''_\Phi v_0\|_{\mathcal{B}^{2l}}. \tag{3.7}$$

But $\widetilde{g} - d''_\Phi v_0 = \Phi \lrcorner \partial_b A_1 \widetilde{g} + A_2(\tau_\Phi(\widetilde{g}))$ and hence $\|\widetilde{g} - d''_\Phi v_0\|_{\mathcal{B}^{2l}} \leq \|\Phi \lrcorner \partial_b\|\|A_1 \widetilde{g}\|_{\mathcal{B}^{2l+1}} + \epsilon\|\widetilde{g}\|_{\mathcal{B}^{2l}}$. This implies

$$\|\widetilde{g} - d''_\Phi v_{n+1}\|_{\mathcal{B}^{2l}} \leq (n+2)\epsilon^{n+1}\|\Phi \lrcorner \partial_b\|\|A_1\|\|\widetilde{g}\|_{\mathcal{B}^{2l+1}} + \epsilon^{n+2}\|\widetilde{g}\|_{\mathcal{B}^{2l}}. \tag{3.8}$$

Since $\epsilon < 1$, the righthand side of (3.8) tends to zero, when n tends to infinity and by continuity of the operator d''_Φ from $\mathcal{B}^{2l+1}(\mathbb{M}, E)$ into $\mathcal{B}_{0,1}^{2l}(\mathbb{M}, E)$, we get that v is a solution of (3.1).

3.3   Solution of the embedding problem

In the setting of the beginning of section 2, let us consider the following anisotropic Hölder spaces of functions:

- $\mathcal{A}^\alpha(M_0)$, $0 < \alpha < 1$, is the set of continuous functions on M_0 which are in $\mathcal{C}^{\alpha/2}(M_0)$.
- $\mathcal{A}^{1+\alpha}(M_0)$, $0 < \alpha < 1$, is the set of functions f such that $f \in \mathcal{C}^{(1+\alpha)/2}(M_0)$ and $X_\mathbb{C}f \in \mathcal{C}^{\alpha/2}(M_0)$, for all complex tangent vector fields $X_\mathbb{C}$ to M_0. Set

$$\|f\|_{A\alpha} = \|f\|_{(1+\alpha)/2} + \sup_{\|X_\mathbb{C}\| \leq 1} \|X_\mathbb{C}f\|_{\alpha/2} \tag{3.9}$$

- $\mathcal{A}^{l+\alpha}(M_0)$, $l \geq 2$, $0 < \alpha < 1$, is the set of functions f of class $\mathcal{C}^{[l/2]}$ such that $Xf \in \mathcal{A}^{l-2+\alpha}(M_0)$, for all tangent vector fields X to M and $X_\mathbb{C}f \in \mathcal{A}^{l-1+\alpha}(M_0)$, for all complex tangent vector fields $X_\mathbb{C}$ to M_0.

Fix some $0 < \alpha < 1$ and set $\mathcal{B}^l(M_0) = \mathcal{A}^{l+\alpha}(M_0)$. This sequence $(\mathcal{B}^l(M_0), l \in \mathbb{N})$ is a sequence of Banach spaces which satisfies properties (i) to (iv) listed in section 3.1

Moreover it is proved in [9] that the operators A_r, $r = 1, 2$, from Theorem 3.3 are linear continuous operators between the anisotropic Hölder spaces $\mathcal{B}_{0,r}^l(M_0, E)$ and $\mathcal{B}_{0,r-1}^{l+1}(M_0, E)$.

One can also consider the anisotropic Hölder spaces introduced by Folland and Stein when they studied the tangential Cauchy-Riemann complex on the Heisenberg group and more generally on strictly pseudoconvex CR manifolds.

Let M be a generic CR manifold of class $\mathcal{C}^\infty$ of real dimension $2n$ and CR dimension $2n - k$ and D be a relatively compact domain in M. Let $X_1, \ldots, X_{2n-2k}$ be a real basis of HM. A $\mathcal{C}^1$ curve $\gamma : [0, r] \to M$ is called admissible if for every $t \in [0, r]$,

$$\frac{d\gamma}{dt}(t) = \sum_{j=1}^{2n-2k} c_j(t) X_j(\gamma(t))$$

where $\sum |c_j(t)|^2 \leq 1$.

The Folland-Stein anisotropic Hölder spaces $\Gamma^{p+\alpha}(\overline{D} \cap M)$ are defined in the following way:

- $\Gamma^\alpha(\overline{D} \cap M)$, $0 < \alpha < 1$, is the set of continuous fonctions in $\overline{D} \cap M$ such that if for every $x_0 \in \overline{D} \cap M$

$$\sup_{\gamma(.)} \frac{|f(\gamma(t) - f(x_0)|}{|t|^\alpha} < \infty$$

for any admissible complex tangent curve γ through x_0.

- $\Gamma^{p+\alpha}(\overline{D} \cap M)$, $p \geq 1$, $0 < \alpha < 1$, is the set of continuous fonctions in M such that $X_\mathbb{C} f \in \Gamma^{p-1+\alpha}(\overline{D} \cap M)$, for all complex tangent vector fields $X_\mathbb{C}$ to M.

Fix some $0 < \alpha < 1$, the sequence $(\Gamma^{p+\alpha}, p \in \mathbb{N})$ is also a sequence of Banach spaces which satisfies properties (i) to (iv) listed in section 3.1

Continuity properties for the operators A_r and B_r, $r = 1, 2$, defined in Theorem 3.3 are proved in [9]. More precisely, for all $p \in \mathbb{N}$ and $0 < \alpha < 1$, the operators A_r, $r = 1, 2$, from Theorem 3.3 are continuous from $\Gamma_{0,r}^{p+\alpha}(M)$ into $\Gamma_{0,r-1}^{p+1+\alpha}(M)$.

We can apply now the method developed in section 3.2 with $E = T_{1,0}X_{|M_0}$ to solve the equation

$$\overline{\partial}_b^\Phi \sigma = -\overline{\partial}_b^\Phi Id$$

and a unique solution is given by a fixed point σ of the map Θ if $\|\Phi\|_{l+3}$ is sufficiently small, moreover this fixed point is contained in the neighborhood U of the zero section in TX on which the exponential map is defined, once again if $\|\Phi\|_{l+3}$ is small enough, since $\|\sigma\|_l < \|\Phi\|_{l+2}$. We deduce that $\mathcal{E} = F \circ \mathcal{E}_0$, with F defined by $F(x) = \exp_x \sigma(x)$, is the embedding we are looking for, which ends the proof of Theorem 3.1.

4 Stability of embeddability in $\mathbb{C}P^N$

In this section we will consider the case when the compact CR manifold $(\mathbb{M}, H_{0,1}\mathbb{M})$ is embeddable in some $\mathbb{C}P^N$. Let $\mathcal{E}_0 : \mathbb{M} \to M_0 \subset \mathbb{C}P^N$ be a $\mathcal{C}^\infty$-smooth CR embedding, it is defined by some homogeneous coordinates $(f_1, \ldots, f_N)$, each f_j, $j = 1, \ldots, N$, being

a CR function such that on the set $U_j = \{x \in \mathbb{M} \mid f_j \neq 0\}$, $(\frac{f_1}{f_j}, \ldots, \frac{f_{j-1}}{f_j}, \frac{f_{j+1}}{f_j}, \ldots, \frac{f_N}{f_j})$ defines a diffeomorphism.

Let us consider an horizontal perturbation $\widehat{H}_{0,1}\mathbb{M}$ of $H_{0,1}\mathbb{M}$, we are looking for a CR embedding $\mathcal{E}$ of $(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ in $\mathbb{C}P^N$. As we will consider only small perturbations of the original structure, we are lead to look for some $\mathcal{E}$ given by some small perturbation $(f_1 - g_1, \ldots, f_N - g_N)$ of the original homogeneous coordinates $(f_1, \ldots, f_N)$, which have to satisfy $\overline{\partial}_b^\Phi(f_j - g_j) = 0$ for all $j = 1, \ldots, N$. Using the definition of the operator $\overline{\partial}_b^\Phi$, this is equivalent to

$$\overline{\partial}_b^\Phi g_j = \Phi \lrcorner \partial_b f_j \quad \text{for all } j = 1, \ldots, N. \tag{4.1}$$

Note that the second member of these equations is controlled by the form Φ which defines the perturbation, so that if we can solve the equation $\overline{\partial}_b^\Phi g = f$ with $\mathcal{C}^l$-estimates uniformly with respect to Φ, then for Φ sufficiently small the homogeneous coordinates $(f_1 - g_1, \ldots, f_N - g_N)$ will define the CR embedding we are looking for.

4.1 Solving the $\overline{\partial}_b^\Phi$ with estimates

We are interested in solving the equation

$$\overline{\partial}_b^\Phi \alpha = \beta \tag{4.2}$$

with estimates, uniformly with respect to Φ, in bidegree $(0,1)$ for an horizontal perturbation, given by a form Φ, of the CR structure of an abstract compact CR manifold $(\mathbb{M}, H_{0,1}\mathbb{M})$ of class $\mathcal{C}^\infty$, when Φ is sufficiently small. We will follow the method used by [13] leading to a strong Hodge decomposition theorem and homotopy formulas.

We equip $\mathbb{M}$ with a hermitian metric such that $H_{0,1}\mathbb{M}$ and $H_{1,0}\mathbb{M}$ are orthogonal and we consider the Kohn-Laplacian

$$\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b.$$

For $\mathbb{M}$ equipped with the perturbed structure $\widehat{H}_{0,1}\mathbb{M}$, we can also consider the Kohn-Laplacian

$$\Box_b^\Phi = \overline{\partial}_b^\Phi \overline{\partial}_b^{\Phi*} + \overline{\partial}_b^{\Phi*} \overline{\partial}_b^\Phi$$

replacing the $\overline{\partial}_b$-operator by the new operator $\overline{\partial}_b^\Phi$. We first establish some *a priori* estimates for $\Box_b^\Phi$.

Theorem 4.1. *Let $(\mathbb{M}, H_{0,1}\mathbb{M})$ be an abstract compact CR manifold of class $\mathcal{C}^\infty$ and $\widehat{H}_{0,1}\mathbb{M}$ be an horizontal perturbation of $H_{0,1}\mathbb{M}$ defined by a $(0,1)$-form $\Phi \in \mathcal{C}_{0,1}^l(\mathbb{M}, H_{1,0}\mathbb{M})$, $l \geq 1$, such that $\|\Phi\|_{\mathcal{C}^0} < 1$. Assume that the Levi form of $\mathbb{M}$ satisfies condition $Y(1)$ at $x_0 \in \mathbb{M}$. Then there exists $\delta > 0$ and a sufficiently small neighborhood U of x_0 such that, if $\|\Phi\|_{\mathcal{C}^1} < \delta$,*

$$\|\alpha\|_{\frac{1}{2}}^2 \leq C(1 + K\|\Phi\|_{\mathcal{C}^1})(\|\overline{\partial}_b^\Phi \alpha\|^2 + \|\overline{\partial}_b^{\Phi*} \alpha\|^2 + \|\alpha\|^2)$$

for all $\alpha \in \mathcal{D}_{0,1}(U)$, where $\|.\|$ and $\|.\|_s$ are respectively the L^2 and the Sobolev norms.

Proof. Let $\overline{L}_1, \ldots, \overline{L}_{n-k}$ be an orthonormal basis for $H_{0,1}\mathbb{M}$ and $\overline{\omega}_1, \ldots, \overline{\omega}_{n-k}$ its dual basis. Since $\|\Phi\|_{\mathcal{C}^0} < 1$, then $\overline{L}_1^\Phi, \ldots, \overline{L}_{n-k}^\Phi, L_1^\Phi, \ldots, L_{n-k}^\Phi$, where $L_j^\Phi = L_j - \overline{\Phi(\overline{L}_j)}$, $j = 1, \ldots, n - k$, defines a basis of $H\mathbb{M}$. We denote by $\overline{\omega}_1^\Phi, \ldots, \overline{\omega}_{n-k}^\Phi$ the dual basis of $\overline{L}_1^\Phi, \ldots, \overline{L}_{n-k}^\Phi$, then, for all $j = 1, \ldots, n - k$, we have $\overline{\omega}_j^\Phi = \overline{\omega}_j + ({}^t(I - \Phi)^{-1} - I)(\overline{\omega}_j) = \overline{\omega}_j + \theta_j^\Phi$.

If f is a smooth function on $\mathbb{M}$, we set

$$\|f\|_{\mathcal{L}}^2 = \sum_{i=1}^{n-k} \|L_i f\|^2 + \|f\|^2, \quad \|f\|_{\overline{\mathcal{L}}}^2 = \sum_{i=1}^{n-k} \|\overline{L}_i f\|^2 + \|f\|^2.$$

Let α be a $(0, r)$-form for the new structure $\widehat{H}_{0,1}\mathbb{M}$ with smooth coefficients, then

$$\alpha = \sum_{|I|=r} \alpha_I \, \overline{\omega}_I^\Phi,$$

where $\overline{\omega}_I^\Phi = \overline{\omega}_{i_1}^\Phi \wedge \cdots \wedge \overline{\omega}_{i_r}^\Phi$ if I is the multi-indice $I = (i_1, \ldots, i_r)$ and we have

$$\overline{\partial}_b^\Phi \alpha = \sum_{j=1}^{n-k} \sum_{|I|=r} \overline{L}_j^\Phi \, \alpha_I \, \overline{\omega}_j^\Phi \wedge \overline{\omega}_I^\Phi + \ldots,$$

where $\ldots$ denote terms which do not involve the derivatives of α.

If $r = 0$, then

$$\overline{\partial}_b^\Phi \alpha = \sum_{j=1}^{n-k} (\overline{L}_j - \Phi(\overline{L}_j))\alpha \, \overline{\omega}_j^\Phi$$

$$= \overline{\partial}_b \alpha + \sum_{j=1}^{n-k} [\overline{L}_j \, \alpha \, \overline{\theta}_j^\Phi - \Phi(\overline{L}_j) \, \alpha \, \overline{\omega}_j^\Phi]$$

and since the perturbation is horizontal, i.e. $\Phi \in \mathcal{C}_{0,1}^l(\mathbb{M}, H_{1,0}\mathbb{M})$, we get

$$\|\overline{\partial}_b \alpha\|^2 \leq \|\overline{\partial}_b^\Phi \alpha\|^2 + O(\|\Phi\|_{\mathcal{C}^0}(\|\alpha\|_{\mathcal{L}}^2 + \|\alpha\|_{\overline{\mathcal{L}}}^2)).$$

If $r = 1$, then

$$\overline{\partial}_b^\Phi \alpha = \sum_{i,j=1}^{n-k} (\overline{L}_j - \Phi(\overline{L}_j))\alpha_i \, \overline{\omega}_j^\Phi \wedge \overline{\omega}_i^\Phi + \ldots$$

$$= \overline{\partial}_b \alpha + \sum_{i,j=1}^{n-k} (\overline{L}_j \, \alpha_i \, \overline{\theta}_j^\Phi \wedge \overline{\omega}_i^\Phi - \Phi(\overline{L}_j) \, \alpha_i \, \overline{\omega}_j^\Phi \wedge \overline{\omega}_i^\Phi) + R(\Phi, \alpha),$$

where $R(\Phi, \alpha)$ is controlled by $(1 + \|\Phi\|_{\mathcal{C}^1})\|\alpha\|$, and if $(\overline{L}_j^\Phi)^*$ denotes the Hilbert adjoint

of $\overline{L}_j^{\Phi}$

$$\overline{\partial}_b^{\Phi*}\alpha = \sum_{i,j=1}^{n-k}(\delta_i^j + \lambda_{ij}(\Phi))(\overline{L}_j^{\Phi})^*\alpha_i + R^*(\Phi,\alpha)$$

$$= \sum_{i,j=1}^{n-k}(\delta_i^j + \lambda_{ij}(\Phi))(-L_j - (\Phi(\overline{L}_j))^*)\alpha_i + R^*(\Phi,\alpha)$$

$$= \overline{\partial}_b^*\alpha - \sum_{i,j=1}^{n-k}(\Phi(\overline{L}_j))^* \alpha_j + \lambda_{ij}(\Phi)(-L_j - (\Phi(\overline{L}_j))^*)\alpha_i + R^*(\Phi,\alpha),$$

where $\lambda_{ij}(\Phi)$ is controlled by $\|\Phi\|_{\mathcal{C}^0}$, $\|(\Phi(\overline{L}_j))^*f\| = O(\|\Phi\|_{\mathcal{C}^1}\|f\|_{\overline{\mathcal{L}}})$ and $R^*(\Phi,\alpha)$ is controlled by $\|\Phi\|_{\mathcal{C}^1}\|\alpha\|$.

All this implies

$$\|\overline{\partial}_b\alpha\|^2 \leq \|\overline{\partial}_b^{\Phi}\alpha\|^2 + O(\|\Phi\|_{\mathcal{C}^0}(\|\alpha\|_{\mathcal{L}}^2 + \|\alpha\|_{\overline{\mathcal{L}}}^2)) + O((1 + \|\Phi\|_{\mathcal{C}^1})^2\|\alpha\|^2)$$

and

$$\|\overline{\partial}_b^*\alpha\|^2 \leq \|\overline{\partial}_b^{\Phi*}\alpha\|^2 + O(\|\Phi\|_{\mathcal{C}^1}(\|\alpha\|_{\mathcal{L}}^2 + \|\alpha\|_{\overline{\mathcal{L}}}^2)) + O(\|\Phi\|_{\mathcal{C}^1}^2\|\alpha\|^2).$$

The a *priori* estimates proved in Theorem 3.1 of [13] gives the existence of a constant C_0 such that for any $\alpha \in \mathcal{C}_{0,r}^{\infty}(\mathbb{M})$, $r = 0,1$, the following estimate is satisfied

$$\|\alpha\|_{\mathcal{L}}^2 + \|\alpha\|_{\overline{\mathcal{L}}}^2 \leq C_0(\|\overline{\partial}_b\alpha\|^2 + \|\overline{\partial}_b^*\alpha\|^2 + \|\alpha\|^2).$$

Using the previous calculations we get the existence of two constants C and C' such that

$$\|\alpha\|_{\mathcal{L}}^2 + \|\alpha\|_{\overline{\mathcal{L}}}^2 \leq C(\|\overline{\partial}_b^{\Phi}\alpha\|^2 + \|\overline{\partial}_b^{\Phi*}\alpha\|^2 + \|\alpha\|^2) + C'\|\Phi\|_{\mathcal{C}^1}(\|\alpha\|_{\mathcal{L}}^2 + \|\alpha\|_{\overline{\mathcal{L}}}^2 + \|\alpha\|^2).$$

Choose $\delta < 1$ sufficiently small such that $C'\delta < 1$, then if $\|\Phi\|_{\mathcal{C}^1} < \delta$ there exists a constant C'' such that

$$\|\alpha\|_{\mathcal{L}}^2 + \|\alpha\|_{\overline{\mathcal{L}}}^2 \leq C''(1 + C'\|\Phi\|_{\mathcal{C}^1})(\|\overline{\partial}_b^{\Phi}\alpha\|^2 + \|\overline{\partial}_b^{\Phi*}\alpha\|^2 + \|\alpha\|^2).$$

This implies the theorem since under the condition Y(1) the real and the imaginary part of the vector fields L_j satisfy Hörmander's finite type condition of type 2. $\square$

In the spirit of the pionnier works by Kohn-Nirenberg and Folland-Kohn, following the methods used in section 3 of [13], the a *priori* estimates obtained in Theorem 4.1 gives the existence and the regularity of solutions for the $\square_b^{\Phi}$ and the $\overline{\partial}_b^{\Phi}$ equations on compact CR manifolds. Denote by

$$\mathcal{H}_{0,1}^{\Phi}(\mathbb{M}) = \ker \square_b^{\Phi} = \{\alpha \in L_{0,1}^2(\mathbb{M}) \mid \overline{\partial}_b^{\Phi}\alpha = \overline{\partial}_b^{\Phi*}\alpha = 0\}$$

and by H_b^{Φ} the projection operator from $L_{0,1}^2$ onto $\mathcal{H}_{0,1}^{\Phi}(\mathbb{M})$. We have the following strong Hodge decomposition

Theorem 4.2. *Let* $(\mathbb{M}, H_{0,1}\mathbb{M})$ *be an abstract compact CR manifold of class* $\mathcal{C}^\infty$ *and* $\widehat{H}_{0,1}\mathbb{M}$ *be an horizontal perturbation of* $H_{0,1}\mathbb{M}$ *defined by a* $(0,1)$-*form* $\Phi \in \mathcal{C}^l_{0,1}(\mathbb{M}, H_{1,0}\mathbb{M})$, $l \geq 1$. *Suppose that the Levi form of* $\mathbb{M}$ *satisfies condition* $Y(1)$ *and* $\|\Phi\|_{\mathcal{C}^1}$ *is sufficiently small, then there exists a compact operator* $G^\Phi_b : L^2_{0,r}(\mathbb{M}) \to Dom(\Box^\Phi_b)$, $r = 0, 1$ *such that*

(i) For all $s \in \mathbb{N}$ *and* $r = 0, 1$, G^Φ_b *is continuous from* $W^s_{0,r}(\mathbb{M})$ *into* $W^{s+1}_{0,r}(\mathbb{M})$, *more precisely there exists* $\delta > 0$ *such that, if* $\|\Phi\|_{\mathcal{C}^{2+s}} < \delta$, *there exists a constant* C_s *independent of* Φ *such that*

$$\|G^\Phi_b(\alpha)\|_{s+1} \leq C_s \|\alpha\|_s,$$

for $\alpha \in W^s_{0,r}(\mathbb{M})$.

(ii) For any $f \in L^2(\mathbb{M})$

$$f = \overline{\partial}^{\Phi*}_b \overline{\partial}^\Phi_b G^\Phi_b f + H^\Phi_b f,$$

and for any $\alpha \in L^2_{0,1}(\mathbb{M})$

$$\alpha = \overline{\partial}^\Phi_b \overline{\partial}^{\Phi*}_b G^\Phi_b \alpha + \overline{\partial}^{\Phi*}_b \overline{\partial}^\Phi_b G^\Phi_b \alpha + H^\Phi_b \alpha.$$

(iii) $G^\Phi_b H^\Phi_b = H^\Phi_b G^\Phi_b = 0$, $G^\Phi_b \Box^\Phi_b = \Box^\Phi_b G^\Phi_b = I - H^\Phi_b$ *on* $Dom(\Box^\Phi_b)$ *and if* G^Φ_b *is also defined on* $L^2_{0,2}(\mathbb{M})$ *(respectively* $L^2_{0,0}(\mathbb{M})$*),* $G^\Phi_b \overline{\partial}^\Phi_b = \overline{\partial}^\Phi_b G^\Phi_b$ *on* $Dom(\overline{\partial}^\Phi_b)$ *(respectively* $G^\Phi_b \overline{\partial}^{\Phi*}_b = \overline{\partial}^{\Phi*}_b G^\Phi_b$ *on* $Dom(\overline{\partial}^{\Phi*}_b)$*).*

From this theorem we can deduce some results on the solvability of (4.2).

Corollary 4.3. *Under the hypotheses of Theorem 4.2, for any* $f \in L^2(\mathbb{M})$

$$\overline{\partial}^\Phi_b f = \overline{\partial}^\Phi_b \overline{\partial}^{\Phi*}_b G^\Phi_b \overline{\partial}^\Phi_b f.$$

Moreover there exists $\delta > 0$ *such that, if* $\|\Phi\|_{\mathcal{C}^{2+s}} < \delta$, *there exists a constant* C_s *independent of* Φ *such that*

$$\|\overline{\partial}^{\Phi*}_b G^\Phi_b \overline{\partial}^\Phi_b f\|_{s+\frac{1}{2}} \leq C_s \|\overline{\partial}^\Phi_b f\|_s,$$

Proof. We apply Theorem 4.2 in degree 0. If $f \in L^2(\mathbb{M})$ we get

$$f = \overline{\partial}^{\Phi*}_b \overline{\partial}^\Phi_b G^\Phi_b f + H^\Phi_b f,$$

which implies

$$\overline{\partial}^\Phi_b f = \overline{\partial}^\Phi_b \overline{\partial}^{\Phi*}_b \overline{\partial}^\Phi_b G^\Phi_b f + \overline{\partial}^\Phi_b H^\Phi_b f.$$

Using that $\ker \Box^\Phi_b \subset \ker \overline{\partial}^\Phi_b$ and that G^Φ_b exists in bidegree $(0,1)$ and commute with $\overline{\partial}^\Phi_b$ since $\mathbb{M}$ satisfies condition $Y(1)$ we obtain

$$\overline{\partial}^\Phi_b f = \overline{\partial}^\Phi_b \overline{\partial}^{\Phi*}_b G^\Phi_b \overline{\partial}^\Phi_b f.$$

The estimate can be deduced in a classical way from the estimate (i) in Theorem 4.2, see e.g. Theorem 8.4.14 in [3]. $\qquad\square$

4.2 Stability theorem

Let us go back to the setting of the beginning of Section 4. Note that if the CR manifold $(\mathbb{M}, H_{0,1}\mathbb{M})$ is 2-concave its Levi form satisfies condition Y(1) and so we can apply the previous results on the solvability of the $\overline{\partial}_b^\Phi$-equation to the equations

$$\overline{\partial}_b^\Phi g_j = \overline{\partial}_b^\Phi f_j = \Phi \lrcorner \partial_b f_j \quad \text{for all } j = 1, \dots, N. \tag{4.3}$$

Assume that $\|\Phi\|_{\mathcal{C}^{l+2}}$ is sufficiently small to apply Corollary 4.3, then there exists a constant C independent of Φ such that if g_j satisfies (4.3) then

$$\|g_j\|_{\mathcal{C}^l} \le C\|\Phi\|_{\mathcal{C}^l}\|f_j\|_{\mathcal{C}^{l+1}},$$

and taking Φ even smaller $(f_1 - g_1, \dots, f_N - g_N)$ will define homogeneous CR coordinates on $(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$. So we have proved

Theorem 4.4. *Let* $(\mathbb{M}, H_{0,1}\mathbb{M})$ *be an abstract compact CR manifold of class* $\mathcal{C}^\infty$, *which is smoothly embeddable as a CR manifold in* $\mathbb{C}P^n$. *Assume that* $(\mathbb{M}, H_{0,1}\mathbb{M})$ *is 2-concave and let* $\widehat{H}_{0,1}\mathbb{M}$ *be an horizontal perturbation of* $H_{0,1}\mathbb{M}$ *defined by a* $(0,1)$-*form* $\Phi \in \mathcal{C}_{0,1}^{l+2}(\mathbb{M}, H_{1,0}\mathbb{M})$, $l \ge 1$.

Then there exists a positive real number δ *such that if* $\|\Phi\|_{\mathcal{C}^{l+2}} < \delta$, *then the CR manifold* $(\mathbb{M}, \widehat{H}_{0,1}\mathbb{M})$ *is embeddable in* $\mathbb{C}P^n$ *as a CR submanifold of class* $\mathcal{C}^l$.

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